Name of College - S. S. College, J. Bad
Deptt - Mathematics
Topic - Successive Differentiation
class - B. Sc I (HONS)
Date -

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Successive Differentiation

Notation → If $y = f(x) \Rightarrow \frac{dy}{dx}$ or $f'(x)$ or y_1

Expression $\frac{dy}{dx}$ is the first derivative of the function $y = f(x)$

If $\frac{dy}{dx}$ is differentiated once again then it is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ or y_2

the Expression $\frac{d^2y}{dx^2}$ is called second derivative of the function $y = f(x)$

In the same manner

$\frac{d^3y}{dx^3}$ is called the third derivative of $y=f(x)$

In General $\frac{d^ny}{dx^n}$ is called the nth derivative of $F(x)$

nth derivative of some standard Functions

1. If $y = x^m$ then find y_n

Since $y = x^m$

$$\Rightarrow y_1 = m(x)^{m-1} = m x^{m-1}$$

$$y_2 = m(m-1) x^{m-2}$$

$$y_3 = m(m-1)(m-2) x^{m-3}$$

$$\cdots\cdots$$

$$y_n = m(m-1)(m-2)\cdots(m-n+1) x^{m-n}$$

thus if $y = x^m$

$$\Rightarrow y_n = m(m-1)(m-2)\cdots(m-n+1) x^{m-n}$$

If $y = x^n$

$$\Rightarrow y_n = n(n-1)(n-2)\cdots\cdots 1 \cdot x^0$$

$$= n(n-1)(n-2)\cdots 2\cdot 1$$

$$= \lfloor n \quad \Rightarrow y_{n+1} = 0 \quad y_{n+2} = 0 \cdots \text{etc.}$$

Cor:→ If $y = (ax+b)^n$ Find y_n

Since $y = (ax+b)^n$

$y_1 = na(ax+b)^{n-1}$

$y_2 = a^2 n(n-1)(ax+b)^{n-2}$

$y_3 = a^3 n(n-1)(n-2)(ax+b)^{n-3}$

$\vdots$

$y_n = a^n n(n-1)(n-2)\cdots 1.$

$= a^n \lfloor n$

$\Rightarrow y_{n+1} = 0$

2. If $y = \frac{1}{ax+b}$ then Find y_n

Here $y = \frac{1}{ax+b} = (ax+b)^{-1}$

$\Rightarrow y_1 = (-1)a(ax+b)^{-2}$

$\Rightarrow y_2 = (-1)^2 a^2 1.2 (ax+b)^{-3}$

$\Rightarrow y_3 = (-1)^3 a^3 1.2.3 (ax+b)^{-4}$

$\Rightarrow y_n = (-1)^n a^n 1.2.3 \cdots n (ax+b)^{-(n+1)}$

$= \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}$

Thus If $y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n a^n \lfloor n}{(ax+b)^{n+1}}$

3. If $y = e^{mx}$ then Find y_n.

Here $y = e^{mx}$

$\Rightarrow y_1 = m e^{mx}$

$\Rightarrow y_2 = m^2 e^{mx}$

$\Rightarrow y_3 = m^3 e^{mx}$

$\vdots$

$y_n = m^n e^{mx}$

Thus $y = e^{mx} \Rightarrow y_n = m^n e^{mx}$

IV If $y = \log(ax+b)$ then find y_n

Here $y = \log(ax+b)$

$\Rightarrow y_1 = \frac{1}{ax+b} \cdot a = a \cdot (ax+b)^{-1}$

$\Rightarrow y_2 = a^2 (-1) (ax+b)^{-2}$

$\Rightarrow y_3 = a^3 (-1)^2 \, 1.2 \, (ax+b)^{-3}$

$\therefore y_n = a^n (-1)^{n-1} \, 1 \cdot 2 \cdot 3 \cdots (n-1) \, (ax+b)^{-n}$

$= \frac{(-1)^{n-1} a^n (n-1}{(ax+b)^n}$

V If $y = \sin(ax+b)$ then Find y_n

Here $y = \sin(ax+b)$

$\Rightarrow y_1 = a \cos(ax+b)$

$= a \sin[\pi/2 + ax+b]$

$\Rightarrow y_2 = a^2 \cos[\pi/2 + ax+b]$

$= a^2 \sin[2 \cdot \pi/2 + ax+b]$

$\therefore y_n = a^n \sin[n\frac{\pi}{2} + ax+b]$

VI If $y = \cos(ax+b)$ then Find y_n

Here $y = \cos(ax+b)$

$\Rightarrow y_1 = -a \sin(ax+b)$

$= a \cos[\pi/2 + ax+b]$

$y_2 = a^2 \sin(\pi/2 + ax+b]$.

$= a^2 \cos[\pi/2 + \pi/2 + (ax+b)]$

$= a^2 \cos[2\pi/2 + ax+b]$

$\therefore y_n = a^n \cos[n\pi/2 + ax+b]$

If $y = e^{ax} \sin(bx+c)$ then Find y_n

Here $y = e^{ax} \sin(bx+c)$

$$y_1 = a e^{ax} \sin(bx+c) + b e^{ax} \cos(bx+c)$$

$$= e^{ax} [a \sin(bx+c) + b \cos(bx+c)]$$

$$= e^{ax}\, r \sin(bx+c+\theta)$$

Where $r = \sqrt{a^2+b^2}$

$\tan\theta = b/a$

$\Rightarrow \theta = \tan^{-1}\frac{b}{a}$

$$y_2 = e^{ax} r^2 \sin(bx+c+2\theta)$$

$$y_n = e^{ax} r^n \sin(bx+c+n\theta)$$

thus if $y = e^{an} \sin(bx+c)$

$$\Rightarrow y_n = r^n e^{ax} \sin(bx+c+n\theta)$$

If $y = \frac{1}{x^2+a^2}$ then Find y_n.

Sol^n :→ Here $y = \frac{1}{x^2+a^2}$

$$= \frac{1}{x^2 - i^2 a^2} \qquad (i^2 = -1)$$

$$= \frac{1}{(x-ia)(x+ia)}$$

$$= \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$= \frac{1}{2ia}\left[(x-ia)^{-1} - (x+ia)^{-1}\right]$$

$$y_1 = \frac{1}{2ia}\left[(-1)\cdot(x-ia)^{-2} - (-1)(x+ia)^{-2}\right]$$

$$y_2 = \frac{1}{2ia}\left[(-1)^2\, 1\cdot 2\, (x-ia)^{-3} - (-1)^2 \cdot 1\cdot 2\cdot (x+ia)^{-3}\right]$$

$$\Rightarrow y_2 = \frac{1}{2ia}(-1)^2 \lfloor 2 \left[(x-ia)^{-3} - (x+ia)^{-3}\right]$$

$$\Rightarrow y_n = \frac{1}{2ia}(-1)^n \lfloor n \left[\left[x-ia\right)^{-(n+1)} - (x+ia)^{-(n+1)}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

Put $x = r\cos\theta$ $\Rightarrow x - ia = r(\cos\theta - i\sin\theta)$

$a = r\sin\theta$ $\Rightarrow (x-ia)^{n+1} = r^{n+1}(\cos\theta - i\sin\theta)^{n+1}$

using De Moivre's theorem.

$$= r^{n+1}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$$

Also $(x+ia)^{n+1} = r^{n+1}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{r^{n+1}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]} - \frac{1}{r^{n+1}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\frac{1}{r^{n+1}}\left[\frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta} - \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\frac{1}{r^{n+1}}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n \lfloor n}{\cancel{2}ia}\cdot\frac{1}{r^{n+1}} \times \cancel{2i}\sin(n+1)\theta$$

$$= \frac{(-1)^n \lfloor n}{a\cdot r^{n+1}}\sin(n+1)\theta.$$

$\because a = r\sin\theta$

$\therefore r = \frac{a}{\sin\theta}$

$r^{n+1} = \frac{a^{n+1}}{\sin^{n+1}\theta}$

$$\therefore y_n = \frac{(-1)^n \lfloor n}{a\cdot a^{n+1}}\sin^{n+1}\theta \sin(n+1)\theta$$

$$= \frac{(-1)^n \lfloor n}{a^{n+2}}\sin^{n+1}\theta \sin(n+1)\theta$$

Where $\tan\theta = \frac{a}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{a}{x}\right)$